LESSON 21 - STUDY GUIDE

Abstract. In this lesson we will study carefully the important class of operators that correspond, in Fourier space, to the multiplication of the Fourier coefficients of the function by a fixed sequence. Examples of such operators are the summability operators, derivatives, translations and convolutions. To be able to look at such operators from a general point of view we will do a brief review of distributions on the circle $\mathbb{T}$ and see that all Fourier multiplier operators on the frequency side are convolutions with a fixed distribution kernel on $\mathbb{T}$. We will finish the lesson by proving that all bounded operators from $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$, for any $1 \leq p, q \leq \infty$, that commute with translations, are such convolution, or Fourier multiplier, operators.

1. Fourier multipliers, convolution operators and distributions on $\mathbb{T}$.

Study material: Most of the presentation in this lesson is mine, and does not follow any particular book. Many books do exist that cover distributions, with different levels of detail, but I tried here to do a fast review of the basic facts in very few pages. I can only recommend more detailed presentations, for those who feel like studying the subject more deeply. As usual, Folland's book [1] has a very good and complete presentation of distributions, covering a lot more than the little that is required for this lesson, starting with the theory of topological vector spaces in section section **5.4 - Topological Vector Spaces** from chapter **5 - Elements of Functional Analysis** and continuing later in the book with a full chapter **9 - Elements of Distribution Theory**. Grafakos's book [2] also has a very complete presentation of distributions in $\mathbb{R}^n$ in chapter **2 - Maximal Functions, Fourier Transform and Distributions** although, interestingly, the distributions on the circle $\mathbb{T}$ that we cover here are only to be found in a sequence of exercises at the end of section **4.3 Multipliers, Transference, and Almost Everywhere Convergence**, where he actually asks the reader to prove and develop by himself the basic required facts and results. The last part of this lesson, on the characterization of bounded operators from $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$ that commute with translations as convolutions, or Fourier multiplier operators, follows closely subsection **4.3.1 Multipliers on the Torus**, there.

Most of the operators that we have encountered up to this point have consisted of convolution operators which, as we now know, on the frequency side correspond to multiplication of the Fourier coefficients of the function by a fixed sequence.

The three main summability methods are the first such operators that should immediately come to mind:

Date: May 11, 2020.

Operator	On the circle side $\mathbb{T}$	On the frequency side $\mathbb{Z}$
Partial sums of the Fourier series $f \mapsto S_N[f]$	$f \mapsto D_N * f$	$\hat{f}(n) \mapsto \chi_{[-N,N]}(n)\hat{f}(n)$
Cesàro means $f \mapsto \sigma_N(f)$	$f \mapsto K_N * f$	$\hat{f}(n) \mapsto \chi_{[-N,N]}(n)\left(1 - \frac{\lvert n\rvert}{N+1}\right)\hat{f}(n)$
Abel means $f \mapsto A_r(f)$	$f \mapsto P_r * f$	$\hat{f}(n) \mapsto r^{\lvert n\rvert}\hat{f}(n)$

where $\chi_{[-N,N]}(n)$ here represents the characteristic function/sequence for the integers $-N \leq n \leq N$.

Many other operators also correspond clearly to multiplication on the frequency side, but cannot be written as convolutions of functions on $\mathbb{T}$ in this same manner because the multiplying sequences are not Fourier coefficients of any $L^1(\mathbb{T})$ functions:

- Derivatives: $f \mapsto \frac{d^k}{dt^k} f$ corresponds to $\hat{f}(n) \mapsto (in)^k \hat{f}(n)$.
- Translations: $f \mapsto \tau_h f = f(\cdot - h)$ corresponds to $\hat{f}(n) \mapsto e^{-inh}\hat{f}(n)$.
- Identity: $f \mapsto f$ corresponds to $\hat{f}(n) \mapsto 1 \cdot \hat{f}(n)$.

In these three examples the terms of the sequences by which the Fourier coefficients of f are multiplied on the frequency side do not even decrease to zero. In fact, the first example shows that differentiation operators actually correspond to polynomially increasing sequences, as $|n| \to \infty$. The last of these examples, the identity map, is the clue to explaining all three of them. As we saw, back in Lesson 14, the constant sequence equal to one is the Fourier transform of the Dirac-δ measure at the origin. One can define convolutions of Borel measures and functions on $\mathbb{T}$ as

$$\mu * f(t) = \int_{\mathbb{T}} f(t-s)d\mu(s),$$

from which it easily follows that $f(t) = \int_{\mathbb{T}} f(t-s)d\delta(s) = \delta * f(t)$, showing that the identity operator can still be interpreted as the convolution of f with the object, in this case the Dirac-δ measure, whose Fourier coefficients are the multiplying constant sequence of ones, on the frequency side. And in this sense we then have

$$f \mapsto \mathrm{Id}f = \delta * f \quad \text{corresponds to} \quad \hat{f}(n) \mapsto 1 \cdot \hat{f}(n).$$

So, to properly regard as convolutions all the operators whose Fourier transform corresponds to multiplication by a fixed sequence on the frequency side, one therefore needs to enlarge the set of objects for which we define Fourier coefficients, beyond $L^1(\mathbb{T})$ functions or even Borel measures $\mathcal{M}(\mathbb{T})$. The right framework is the theory of distributions. Without getting very deep into the the details of the theory, let us just do a brief overview of the basic facts that help to clarify the whole picture.

The set of distributions on $\mathbb{T}$ is the (topological) dual of the infinite dimensional vector space $C^\infty(\mathbb{T})$, i.e. the set of linear continuous functionals defined on $C^\infty(\mathbb{T})$. However, $C^\infty(\mathbb{T})$ is not a Banach space for one cannot pick a single norm for which it is complete. Instead of a single norm, what is adequate, in this case, is an infinite sequence of seminorms in order to define a metric with which it is a complete space,

$$\|\phi\|_k = \sup_{t\in\mathbb{T}} \left| \frac{d^k}{dt^k}\phi(t) \right|, \quad \text{for} \quad \phi \in C^\infty(\mathbb{T}), k = 0, 1, 2, 3, \ldots$$

A topological vector space with a metric topology defined by a countable number of seminorms is called a Fréchet space and it shares many of the properties of Banach spaces (see, for example, Folland [1]) in section **5.4 - Topological Vector Spaces** from chapter **5 - Elements of Functional Analysis**). So, in an analogous manner to a Banach space, $C^\infty(\mathbb{T})$ is a Fréchet space and the distributions, usually denoted by $\mathcal{D}'(\mathbb{T})$[1], its dual. In the framework of the theory of distributions it is common to write $\langle u, \phi\rangle$, for the value of the distribution $u \in \mathcal{D}'(\mathbb{T})$ at $\phi \in C^\infty(\mathbb{T})$ and it is also standard to call the functions in $C^\infty(\mathbb{T})$ *test functions* because, from this slightly different point of view, it is through "testing" the values of u with different functions ϕ that one gets to understand its properties. Continuity of the functionals, instead of a single bound like in Banach spaces, now requires a finite sum of seminorms. So a linear functional $u : C^\infty(\mathbb{T}) \to \mathbb{C}$ is a distribution if it is continuous, with continuity now being equivalent to the existence of a constant $C \geq 0$ and an integer $M \in \mathbb{N}$ such that, for all $\phi \in C^\infty(\mathbb{T})$, the following bound holds

$$|\langle u, \phi\rangle| \leq C(\|\phi\|_0 + \|\phi\|_1 + \cdots + \|\phi\|_M), \tag{1.1}$$

and this is also equivalent to sequential continuity in the sense that, if $\{\phi_n\}$ is a sequence of test functions such that all its derivatives converge uniformly on $\mathbb{T}$ to the corresponding derivatives of another test function ϕ, then $\langle u, \phi_n\rangle \to \langle u, \phi\rangle$.

Any function $f \in L^1(\mathbb{T})$ can then be uniquely identified with a distribution $u_f \in \mathcal{D}'(\mathbb{T})$ defined by

$$\langle u_f, \phi\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\phi(t)dt,$$

and, more generally, Borel measures $\mu \in \mathcal{M}(\mathbb{T})$ by the identification

$$\langle u_\mu, \phi\rangle = \int_{\mathbb{T}} \phi(t)d\mu(t).$$

It is an easy exercise in measure theory to check that this identification is injective, so that, in a way, testing the functionals with every $\phi \in C^\infty(\mathbb{T})$ really identifies functions (almost everywhere) or measures uniquely. One can then think of the space of distributions as a much larger set of objects than $L^1(\mathbb{T})$ functions or Borel measures $\mathcal{M}(\mathbb{T})$, such that it includes them as particular subspaces

$$L^1(\mathbb{T}) \subset \mathcal{M}(\mathbb{T}) \subset \mathcal{D}'(\mathbb{T}).$$

Generalizing then the definition of Fourier coefficients for functions $f \in L^1(\mathbb{T})$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt = \langle u_f, e^{-int}\rangle,$$

and measures $\mu \in \mathcal{M}(\mathbb{T})$

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-int}d\mu(t) = \langle u_\mu, e^{-int}\rangle,$$

one naturally defines the Fourier coefficients of a distribution $u \in \mathcal{D}'(\mathbb{T})$ as the values of testing it on the exponentials $e^{-int} \in C^\infty(\mathbb{T})$,

$$\hat{u}(n) = \langle u, e^{-int}\rangle. \tag{1.2}$$

[1]Laurent Schwartz had the habit of denoting the space of C^∞ functions of compact support by $\mathcal{D}$ - on $\mathbb{T}$ all functions have compact support, of course. So, that the space of distributions, being the dual of the space of smooth functions with compact support, became commonly denoted by $\mathcal{D}'$.

For example, if one looks at the Dirac-δ measure at the origin as a distribution

$$\langle \delta, \phi \rangle = \int_{\mathbb{T}} \phi(t) d\delta(t) = \phi(0),$$

then it is simply the functional that assigns the value at the origin of the test function $\phi \in C^\infty(\mathbb{T})$. So that its Fourier coefficients then are

$$\hat{\delta}(n) = \langle \delta, e^{-int} \rangle = e^{-in0} = 1,$$

for all $n \in \mathbb{Z}$, as we had already seen in Lesson 14.

So not only don't Fourier coefficients of distributions decrease to zero, something that we had already seen for the Dirac-δ measure, they now can increase (at most) polynomially due to the bound (1.1) applied to the exponentials in the definition (1.2). The Riemann-Lebesgue convergence to zero of Fourier coefficients is really just a property of $L^1(\mathbb{T})$ functions.

Convergence in the space of distributions is defined as weak* convergence, so that we say that a sequence of distributions $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $\mathcal{D}'(\mathbb{T})$ if for all $\phi \in C^\infty(\mathbb{T})$ we have $\langle u_n, \phi \rangle \to \langle u, \phi \rangle$ as $n \to \infty$. This topology is so weak that things tend to work out extremely well. So, for example, all the summability kernels converge to a Dirac-δ at the origin *including* the Dirichlet kernel, in this weak sense of distributions

$$\langle u_{D_N}, \phi \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} D_N(t)\phi(t)dt = D_N * \phi(0) = \sum_{n=-N}^{N} \hat{\phi}(n) \to \phi(0) = \langle \delta, \phi \rangle,$$

as $N \to \infty$, because we know that we have pointwise convergence of the partial sums of the Fourier series of smooth functions, such as ϕ. And this means exactly that $D_N \to \delta$ in the sense of distributions. The same holds for any other summability kernel, of course. It is an equally simple exercise to show that, in this very weak sense, the partial sums of the Fourier series of any function in $L^1(\mathbb{T})$ converges to f,

$$D_N * f = \sum_{n=-N}^{N} \hat{f}(n) e^{int} \to f, \quad \text{as} \quad N \to \infty \quad \text{in} \quad \mathcal{D}'(\mathbb{T}),$$

which is the same as saying that, for every $\phi \in C^\infty(\mathbb{T})$ we have $\langle u_{D_N * f}, \phi \rangle \to \langle u_f, \phi \rangle$.

To finish this fast overview of the basic definitions and properties of distributions, let us also look at how basic operations on functions, like translations, convolutions or derivatives, are generalized for distributions. The main idea is always to use duality in order to transfer those operations to the test functions.

So, starting with derivatives, observe that if $f \in C^k(\mathbb{T})$ then, integrating by parts, one can write the distribution corresponding to the k-th order derivative of f as

$$\langle u_{\frac{d^k}{dt^k} f}, \phi \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d^k}{dt^k} f(t)\phi(t)dt = \frac{1}{2\pi} \int_{\mathbb{T}} f(t)(-1)^k \frac{d^k}{dt^k} \phi(t)dt = (-1)^k \langle u_f, \frac{d^k}{dt^k} \phi \rangle.$$

The advantage of this dual formula is that, while both sides of the identity correspond to the distribution identified with $\frac{d^k}{dt^k} f$ when $f \in C^k(\mathbb{T})$, the right hand side is always well defined even when f is not differentiable at all, because the k derivative operate on the test function only. So, generalizing this idea, one defines the k-th order derivative of any distribution as

$$\langle \frac{d^k}{dt^k} u, \phi \rangle = (-1)^k \langle u, \frac{d^k}{dt^k} \phi \rangle,$$

making distributions differentiable as many times as desired.

Following the same duality motivation for the convolution, we imagine two functions g and f in $L^1(\mathbb{T})$, with g playing the role of the distribution. Then, we have

$$\langle u_{g*f}, \phi\rangle = \frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)g(s)ds\right)\phi(t)dt = \frac{1}{2\pi}\int_{\mathbb{T}} g(s)\left(\frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)\phi(t)dt\right)ds = \langle u_g, f(-\cdot)*\phi\rangle.$$

Of course $f(-\cdot) * \phi(s) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)\phi(t)dt$ is the convolution of the reflected $L^1(\mathbb{T})$ function $f(-\cdot)$ and the $C^\infty(\mathbb{T})$ function ϕ which, from our knowledge of differentiability of convolutions in Lesson 10, is $C^\infty(\mathbb{T})$. Again, therefore, we can define the convolution of a general distribution $u \in \mathcal{D}'(\mathbb{T})$ and $f \in L^1(\mathbb{T})$ from the right hand side of this formula

$$\langle u * f, \phi\rangle = \langle u, f(-\cdot) * \phi\rangle.$$

Finally, for translations, if we start as before with the duality for distributions corresponding to functions $f \in L^1(\mathbb{T})$

$$\langle u_{\tau_h f}, \phi\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(t-h)\phi(t)dt = \frac{1}{2\pi}\int_{\mathbb{T}} f(s)\phi(s+h)ds = \langle u_f, \tau_{-h}\phi\rangle,$$

so that for general distributions we thus define their translation operator as

$$\langle \tau_h u, \phi\rangle = \langle u, \tau_{-h}\phi\rangle.$$

It is an easy exercise to show that all these three definitions, besides being linear in ϕ, also satisfy the continuity bound (1.1) so that they, indeed are distributions. From these definitions, the properties in the following proposition are just simple exercises, generalizing to distributions several facts that are already known for functions.

Proposition 1.1. *Let $u \in \mathcal{D}'(\mathbb{T})$. Then*

(1) $\tau_h(u * f) = (\tau_h u) * f = u * (\tau_h f)$, *for $f \in L^1(\mathbb{T})$ and $h \in \mathbb{T}$.*
(2) $\frac{d^k}{dt^k}(u * f) = \left(\frac{d^k}{dt^k}u\right) * f = u * \left(\frac{d^k}{dt^k}f\right) = \left(\frac{d^i}{dt^i}u\right) * \left(\frac{d^j}{dt^j}f\right)$, *for $i + j = k$ and $f \in C^k(\mathbb{T})$.*
(3) $\widehat{u * f}(n) = \widehat{u}(n)\widehat{f}(n)$, *for $f \in L^1(\mathbb{T})$.*
(4) $\widehat{\frac{d^k}{dt^k}u}(n) = (in)^k\widehat{u}(n)$.
(5) $\widehat{\tau_h u}(n) = e^{-inh}\widehat{u}(n)$, *for $h \in \mathbb{T}$.*

In particular, when the distribution is the Dirac-δ, we have

$$\frac{d^k}{dt^k}f = \frac{d^k}{dt^k}(\delta * f) = \left(\frac{d^k}{dt^k}\delta\right) * f$$

where the distribution $\frac{d^k}{dt^k}\delta$ is given by

$$\langle\frac{d^k}{dt^k}\delta, \phi\rangle = (-1)^k\langle\delta, \frac{d^k}{dt^k}\phi\rangle = (-1)^k\phi^{(k)}(0),$$

for test functions $\phi \in C^\infty(\mathbb{T})$, and has Fourier coefficients given by

$$\widehat{\frac{d^k}{dt^k}}\delta(n) = \langle\frac{d^k}{dt^k}\delta, e^{-int}\rangle = (-1)^k\frac{d^k e^{-int}}{dt^k}_{|t=0} = (in)^k,$$

which could also have been obtained directly from Property (4) in the previous proposition.

Also, with respect to translation operator

$$\tau_h f = \tau_h(\delta * f) = (\tau_h\delta) * f,$$

where the translation of the Dirac-δ at the origin, $\tau_h\delta$, as expected is simply a Dirac-δ at $t = h$,

$$\langle\tau_h\delta, \phi\rangle = \langle\delta, \tau_{-h}\phi\rangle = \phi(0+h) = \phi(h) = \langle\delta_h, \phi\rangle,$$

and has Fourier coefficients given by

$$\widehat{\tau_h\delta}(n) = \langle \tau_h\delta, e^{-int}\rangle = \langle \delta_h, e^{-int}\rangle = e^{-inh},$$

which again could have been obtained directly from Property (5) in the previous proposition.

We can thus construct an analogous chart as the one at the beginning of this lesson, but now using convolutions with distributions as the more general form of writing operators

Operator	On the circle side $\mathbb{T}$	On the frequency side $\mathbb{Z}$
Derivatives $f \mapsto \frac{d^k}{dt^k} f$	$f \mapsto \frac{d^k\delta}{dt^k} * f$	$\hat{f}(n) \mapsto (in)^k \hat{f}(n)$
Translation $f \mapsto \tau_h f$	$f \mapsto (\tau_h\delta) * f = \delta_h * f$	$\hat{f}(n) \mapsto e^{-inh}\hat{f}(n)$
Identity $f \mapsto \mathrm{Id} f$	$f \mapsto \delta * f$	$\hat{f}(n) \mapsto 1 \cdot \hat{f}(n)$

As mentioned above, due to the continuity bound (1.1) satisfied by any distribution $u \in \mathcal{D}'(\mathbb{T})$ the Fourier coefficients of u grow, at most, polynomially

$$u \in \mathcal{D}'(\mathbb{T}) \quad \Rightarrow \quad \hat{u}(n) = O(|n|^M),$$

for some $M \in \mathbb{N}$. We will call a sequence $\{c_n\}_{n\in\mathbb{Z}}$ *tempered*, or of *tempered growth*, if it satisfies such a polynomial growth bound. The converse also holds.

Theorem 1.2. *Let $\{c_n\}_{n\in\mathbb{Z}}$ be a tempered sequence of complex numbers, $c_n = O(|n|^m)$ for some $m \in \mathbb{N}$. Then, there exists a distribution $u \in \mathcal{D}'(\mathbb{T})$ such that $\hat{u}(n) = c_n$.*

Proof. For test functions $\phi \in C^\infty(\mathbb{T})$ we know that their Fourier coefficients $\widehat{\phi}(n)$ decay faster than any power of n, i.e. $\widehat{\phi}(n) = o(\frac{1}{|n|^k})$, for any $k \in \mathbb{N}$. Therefore, necessarily, the series

$$S_\phi = \sum_{n=-\infty}^{\infty} c_n \widehat{\phi}(-n),$$

converges absolutely. The mapping $\phi \mapsto S_\phi$ is then clearly a linear functional on $C^\infty(\mathbb{T})$. We now want to show that this functional is the desired distribution, only having to prove the continuity bound (1.1) to conclude it. But

$$|S_\phi| \leq |c_0||\widehat{\phi}(0)| + \sum_{|n|\geq 0} |c_n| \frac{|\widehat{\phi^{(m+2)}}(-n)|}{|n|^{m+2}},$$

from the decay properties of Fourier coefficients of differentiable functions, seen in Lesson 14, Proposition 1.2, so that we then have

$$|S_\phi| \leq C(\|\phi\|_{L^1(\mathbb{T})} + \|\phi'\|_{L^1(\mathbb{T})} + \cdots + \|\phi^{(m+2)}\|_{L^1(\mathbb{T})}) \leq C(\|\phi\|_{L^\infty(\mathbb{T})} + \|\phi'\|_{L^\infty(\mathbb{T})} + \cdots + \|\phi^{(m+2)}\|_{L^\infty(\mathbb{T})}),$$

for some $C > 0$, and this is the desired bound, with $M = m + 2$. So we define the distribution $u \in \mathcal{D}'(\mathbb{T})$ as $\langle u, \phi\rangle = S_\phi$ and to finally check that its Fourier coefficients really are the terms of the sequence c_n, we compute

$$\langle u, e^{-int}\rangle = S_{e^{-int}} = \sum_{j=-\infty}^{\infty} c_j \widehat{e^{-int}}(-j) = c_n,$$

thus concluding the proof. □

The following theorem, establishing the convergence in $\mathcal{D}'(\mathbb{T})$ of the partial sums of the Fourier series of $u \in \mathcal{D}'(\mathbb{T})$ to u itself, follows easily from the previous definitions, duality and the very strong convergence of Fourier series of the test functions in $C^\infty(\mathbb{T})$, seen in the convergence theorems of the previous lessons.

Theorem 1.3. *Let $u \in \mathcal{D}'(\mathbb{T})$ be a distribution and consider its Fourier coefficients $\hat{u}(n)$. Then*

$$\sum_{n=-N}^{N} \hat{u}(n) e^{int} \to u,$$

in the sense of distributions, as $N \to \infty$.

And as an immediate corollary we obtain the uniqueness of the Fourier coefficients of distributions.

Corollary 1.4. *Let $u \in \mathcal{D}'(\mathbb{T})$ be such that $\hat{u}(n) = 0$ for all $n \in \mathbb{Z}$. Then $u = 0$.*

Fourier series, in the space of distributions, becomes then a much more symmetrical theory, as we have a full bijection between distributions and tempered sequences on $\mathbb{Z}$, of which $L^1(T)$ functions are just a particular subspace. The drawback, of course, is that the convergence in the sense of distributions is extremely weak, and that is why it works so well. On the other hand it does not tell us much about pointwise or norm convergence of Fourier series for functions, which has been our goal since the beginning.

Getting back to the reason that brought us to this fast overview of distributions, we can now define a convolution operator associated to any fixed tempered sequence used for multiplying the Fourier coefficients of a function.

Definition 1.5. *Let $\{c_n\}_{n\in\mathbb{Z}}$ be a tempered sequence of complex numbers. The* Fourier multiplier *operator associated to it is defined as the linear map that, to each $f \in L^1(\mathbb{T})$, assigns the unique distribution $Tf \in \mathcal{D}'(\mathbb{T})$ with Fourier coefficients given by $\widehat{Tf}(n) = c_n \hat{f}(n)$,*

$$f \in L^1(\mathbb{T}) \mapsto Tf = \mathcal{F}^{-1}(c_n \hat{f}(n)) \in \mathcal{D}'(\mathbb{T}).$$

In simple terms, a Fourier multiplier operator consists of taking the Fourier transform of a function, multiplying it on the frequency side by a fixed sequence, and inverting the result back to what, generally, can only be guaranteed to be a distribution. From the properties seen before, this Fourier multiplier operator can always then be interpreted as a convolution operator with a distributional kernel whose Fourier coefficients correspond to the fixed tempered sequence. So, if $K \in \mathcal{D}'(\mathbb{T})$ is the unique distribution, given by Theorem 1.2 and Corollary 1.4, such that $\widehat{K}(n) = c_n$, then from Property (3) in Proposition 1.1, we finally have the general form, for any $f \in L^1(\mathbb{T})$,

$$Tf = \mathcal{F}^{-1}(c_n \hat{f}(n)) = K * f. \tag{1.3}$$

Of course, in the reverse direction, any convolution operator with a kernel $K \in \mathcal{D}'(\mathbb{T})$ is always a Fourier multiplier operator as well, with the tempered multiplier sequence $\{\widehat{K}(n)\}_{n\in\mathbb{Z}}$.

Theorem 1.6. *There is a bijection between Fourier multiplier operators associated to tempered complex sequences $\{c_n\}_{n\in\mathbb{Z}}$ and convolution operators with kernels $K \in \mathcal{D}'(\mathbb{T})$, by the correspondence $\widehat{K}(n) = c_n$ and* (1.3)*.*

As an important example, it is worth pointing out that polynomial multipliers correspond exactly to differential operators. And, as mentioned before, in Lesson 14, this is one of the reasons that make Fourier analysis such an important tool in the theory of differential equations. So, for instance, for $f \in C^4(\mathbb{T})$ the following differential operator

$$4\frac{d^4 f}{dt^4} - \frac{d^3 f}{dt^3} + 5\frac{d^2 f}{dt^2} + 8\frac{df}{dt} - f = \mathcal{F}^{-1}\left((4n^4 + in^3 - 5n^2 + 8in - 1)\hat{f}(n)\right),$$

corresponds to the Fourier multiplier operator associated to the polynomial sequence $4n^4 + in^3 - 5n^2 + 8in - 1$. This polynomial multiplier sequence is called the Fourier symbol associated to the differential operator.

And if one wants to solve a particular equation, say, for given $g \in C^\infty(\mathbb{T})$, find f such that

$$\frac{d^2 f}{dt^2} - f = g,$$

then, in Fourier space, this problem corresponds to

$$(-n^2 - 1)\hat{f}(n) = \hat{g}(n) \Rightarrow \hat{f}(n) = -\frac{1}{n^2+1}\hat{g}(n),$$

so that the solution is given by the Fourier multiplier operator applied to the right hand side function g, associated to the symbol $-\frac{1}{n^2+1}$,

$$f = \mathcal{F}^{-1}\left(-\frac{1}{n^2+1}\hat{g}(n)\right) = K * g,$$

where the solution kernel $K \in \mathcal{D}'(\mathbb{T})$, corresponding to $K = \mathcal{F}^{-1}\left(-\frac{1}{n^2+1}\right)$, is usually called the *fundamental solution* of the differential equation, and coincides precisely with the solution for the case $g = \delta$.

This collection of ideas, where differential operators and equations on $\mathbb{T}$ are mapped, through the Fourier transform, to Fourier multiplier operators on the frequency side, that can then be studied and solved through the algebraic manipulation of the corresponding symbols, is the root of the modern and extremely powerful theory of symbolic calculus for pseudo-differential operators. A pseudo-differential operator is basically a Fourier multiplier operator with a symbol corresponding to a tempered sequence that, although not exactly a polynomial in frequency - which would correspond then to an exact differential operator - grows, or decays, polynomially as $|n| \to \infty$. So, for example, the differential operator in the previous example is a pseudo-differential operator of order 2 whereas the solution operator, obtained by the convolution with the fundamental solution, is also a pseudo-differential operator, but of order -2.

A very important family of pseudo-differential operators in the theory of differential equations corresponds to the Fourier multiplier operator with symbol $(1+n^2)^{s/2}$, for $s \in \mathbb{R}$, and is, for obvious reasons, denoted often by $(1 - \frac{d^2}{dt^2})^{s/2}$. So that, for $f \in L^1(\mathbb{T})$ we have

$$\left(1 - \frac{d^2}{dt^2}\right)^{\frac{s}{2}} f = \mathcal{F}^{-1}\left((1+n^2)^{\frac{s}{2}}\hat{f}(n)\right),$$

corresponding to generalized derivatives of *any real* order, for $s > 0$, or to integration operators that increase the regularity of functions, for $s < 0$. Through symbolic calculus and Fourier multiplier operators it is thus extremely natural to generalize derivatives to fractional orders. The convolution kernel associated to the pseudo-differential operator $(1 - \frac{d^2}{dt^2})^{s/2}$

$$K_s = \mathcal{F}^{-1}\left((1+n^2)^{\frac{s}{2}}\right),$$

is called the *Bessel potential* of order $-s$.

For the final part of this lesson, let us now see the important connection between Fourier multiplier operators and translations. From Property (1) in Proposition 1.1 we know that translations commute with convolution operators

$$\tau_h(K * f) = K * (\tau_h f),$$

which could also easily be concluded by looking at the corresponding Fourier multipliers

$$\widehat{\tau_h(K * f)}(n) = e^{-inh}(\hat{K}(n)\hat{f}(n)) = \hat{K}(n)(e^{-inh}\hat{f}(n)) = \widehat{K * (\tau_h f)}(n),$$

a simple consequence of the commutativity of the multiplication of the Fourier coefficients on the frequency side. As mention before, in Lesson 10 in the framework of $\mathbb{R}^n$, a crucial fact in harmonic analysis is that this commutativity property actually characterizes convolution operators, and this is the reason why they are so prevalent in the subject. We will now establish the analogue for $\mathbb{T}$, of Theorem 1.2 that was stated in Lesson 10.

Theorem 1.7. *Suppose that $T : L^p(\mathbb{T}) \to L^q(\mathbb{T})$, for some $1 \le p, q \le \infty$, is a linear and continuous operator that commutes with translations. Then, there exists a bounded sequence $\{c_n\}_{n\in\mathbb{Z}}$ such that T corresponds to the Fourier multiplier operator defined by it, $Tf = \mathcal{F}^{-1}(c_n \hat{f}(n))$. Equivalently, there exists a unique distribution kernel $K \in \mathcal{D}'(\mathbb{T})$, with $\hat{K}(n) = c_n$, such that $Tf = K * f$. Besides, $\sup_{n\in\mathbb{Z}} |c_n| \le \|T\|_{L^p(\mathbb{T})\to L^q(\mathbb{T})}$.*

Proof. For $e_n(t) = e^{int}$, regarded as functions in $L^p(\mathbb{T})$, the hypothesis of the theorem yields

$$\tau_h T(e_n) = T(\tau_h e_n) = T(e^{in(\cdot - h)}) = e^{-inh} T(e_n),$$

as functions in $L^q(\mathbb{T})$, i.e. pointwise almost everywhere $t \in \mathbb{T}$ for each $h \in \mathbb{T}$. So that

$$T(e_n)(t) = \tau_{-h} e^{-inh} T(e_n)(t) = e^{-inh} T(e_n)(t+h) = e^{-inh} T(e_n)(h) e^{int},$$

for any fixed h where it holds almost everywhere in $t \in \mathbb{T}$. Denoting $e^{-inh}T(e_n)(h)$ by c_n, we then have

$$T(e_n)(t) = c_n e^{int}, \tag{1.4}$$

confirming that for $\phi \in C^\infty(\mathbb{T})$, where $\phi(t) = \sum_{-\infty}^{\infty} \hat{\phi}(n) e^{int}$, the convergence of the Fourier series holding absolutely and therefore also in all $L^p(\mathbb{T})$, we have

$$T\phi = \sum_{-\infty}^{\infty} \hat{\phi}(n) T(e^{int}) = \sum_{-\infty}^{\infty} c_n \hat{\phi}(n) e^{int},$$

this last convergence holding in $L^q(\mathbb{T})$. So we conclude that, for $\phi \in C^\infty(\mathbb{T})$ we have $\widehat{T\phi} = c_n \hat{\phi}(n)$ and, by density, the same is true for any $f \in L^p(\mathbb{T})$. T is thus a Fourier multiplier operator associated to the sequence $\{c_n\}_{n\in\mathbb{Z}}$ and from (1.4) we get that $|c_n| \le \|T\|_{L^p(\mathbb{T})\to L^q(\mathbb{T})}$, concluding the proof. □

Definition 1.8. *Let $1 \le p, q \le \infty$. A bounded sequence $\{c_n\}_{n\in\mathbb{Z}}$ is called a (L^p, L^q) multiplier if the corresponding Fourier multiplier operator is bounded from $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$. The set of (L^p, L^q) multipliers is denoted by $\mathcal{M}_{p,q}$. When $p = q$ the corresponding multipliers are just called L^p multipliers and the corresponding set is denoted by $\mathcal{M}_p$.*

To conclude this lesson, let us just look at two important specific examples.

Theorem 1.9. *Continuous linear operators $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ that commute with translations correspond precisely to bounded multipliers, i.e. $\mathcal{M}_2 = l^\infty(\mathbb{Z})$. Besides, $\|T\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})} = \|\{c_n\}\|_{l^\infty(\mathbb{Z})}$.*

Proof. That bounded linear operators from $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, which commute with translations, are Fourier multiplier operators associated to bounded sequences is a consequence of the previous theorem, with $\sup_{n\in\mathbb{Z}} |c_n| = \|\{c_n\}\|_{l^\infty(\mathbb{Z})} \le \|T\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})}$. In the opposite direction, from the Plancherel identity, we have

$$\|Tf\|^2_{L^2(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{Tf}(n)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 |\hat{f}(n)|^2 \le \|\{c_n\}\|^2_{l^\infty(\mathbb{Z})} \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|\{c_n\}\|^2_{l^\infty(\mathbb{Z})} \|f\|^2_{L^2(\mathbb{T})},$$

from which $\|T\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})} \le \|\{c_n\}\|_{l^\infty(\mathbb{Z})}$. And this concludes the proof. □

Multipliers in $L^1(\mathbb{T})$ can also be precisely characterized, as Fourier coefficients of Borel measures.

Theorem 1.10. *Bounded linear operators $T : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ that commute with translations are the convolutions with Borel measures, i.e. $Tf = \mu * f$ for some Borel measure $\mu \in \mathcal{M}(\mathbb{T})$, so that the corresponding Fourier multiplier operator is associated to the sequence of Fourier coefficients of μ,*

$$\widehat{Tf}(n) = \hat{\mu}(n)\hat{f}(n),$$

for all $n \in \mathbb{Z}$. In other words the set of $L^1(\mathbb{T})$ multipliers $\mathcal{M}_1$ is the set of Fourier transforms of Borel measures $\mathcal{F}(\mathcal{M}(\mathbb{T}))$, and $\|T\|_{L^1(\mathbb{T})\to L^1(\mathbb{T})} = \|\mu\|_{\mathcal{M}(\mathbb{T})}$.

Proof. If $Tf = \mu * f$ for a Borel measure then

$$\begin{aligned}\|Tf\|_{L^1(\mathbb{T})} = \frac{1}{2\pi}\int_{\mathbb{T}} |\mu * f(t)|\, dt = \frac{1}{2\pi}\int_{\mathbb{T}} \left|\int_{\mathbb{T}} f(t-s)d\mu(s)\right| dt \\ \leq \int_{\mathbb{T}} \frac{1}{2\pi}\int_{\mathbb{T}} |f(t-s)|dt\, d|\mu|(s) \leq \|f\|_{L^1(\mathbb{T})}\int_{\mathbb{T}} d|\mu|(s) = \|\mu\|_{\mathcal{M}(\mathbb{T})}\|f\|_{L^1(\mathbb{T})},\end{aligned}$$

so that we conclude that it is a bounded operator from $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$ which commutes with translations, because it is a convolution operator, and has operator norm

$$\|T\|_{L^1(\mathbb{T})\to L^1(\mathbb{T})} \leq \|\mu\|_{\mathcal{M}(\mathbb{T})}.$$

The converse direction of the proof is somewhat technical and we will not present it here. Those who are interested should see it, for example, in Grafakos' book [2] in section **4.3.1 - Multipliers on the Torus**. □

References

[1] Gerald B. Folland, *Real Analysis, Modern Techniques and Applications,* 2nd Edition, John Wiley & Sons, 1999.
[2] Loukas Grafakos, *Classical Fourier Analysis,* 3rd Edition, Springer, Graduate Texts in Mathematics 249, 2014.